Lecture 19:

13-11-18

We have considered

$$\vec{y}'(t) = A\vec{y}(t) + P_k(t) = \vec{v}_0 + \vec{v}_1 t + \dots + \vec{v}_k t^k.$$

Guess for particular sol: $\vec{Y}(t) = Q_\ell(t) = \vec{u}_0 + \dots + \vec{u}_\ell t^\ell$.

$$\vec{Y}'(t) = \vec{u}_1 + 2\vec{u}_2 t + \dots + j\vec{u}_j t^{j-1} + \dots + \ell\vec{u}_\ell t^{\ell-1}$$

Claim: if $s =$ alg. mult. of 0 in $P_A(x)$

(if 0 is NOT an eigenvalue $\Rightarrow s = 0$).

then $\vec{Y}(t)$ can be solved with $\ell = k+s$

Plug in: $\vec{Y}'(t) = A\vec{Y}(t) + P_k(t)$

$$(\vec{u}_1 - A\vec{u}_0) + (2\vec{u}_2 - A\vec{u}_1)t + \dots + ((j+1)\vec{u}_{j+1} - A\vec{u}_j)t^j + \dots + A\vec{u}_\ell t^\ell$$

$$= \vec{v}_0 + \vec{v}_1 t + \dots + \vec{v}_k t^k$$

Equation to solve:

$A = QJQ^{-1}$, $\vec{z}_i = Q^{-1}\vec{u}_i$, $\vec{w}_i = Q^{-1}\vec{v}_i$

$$\begin{aligned} A\vec{u}_\ell &= 0 \\ \ell\vec{u}_\ell - A\vec{u}_{\ell-1} &= 0 \\ &\vdots \\ (k+1)\vec{u}_{k+1} - A\vec{u}_k &= \vec{v}_k \\ &\vdots \\ \vec{u}_1 - A\vec{u}_0 &= \vec{v}_0 \end{aligned} \qquad \begin{aligned} J\vec{z}_\ell &= 0 \\ \ell\vec{z}_\ell - J\vec{z}_{\ell-1} &= 0 \\ &\vdots \\ (k+1)\vec{z}_{k+1} - J\vec{z}_k &= \vec{w}_k \\ &\vdots \\ \vec{z}_1 - J\vec{z}_0 &= \vec{w}_0 \end{aligned}$$

Eg 1: Say alg. mult. of 0 for $A = 0$

i.e. J is invertible $\Longrightarrow$ the equation can be solved for $k = l$.

Eg 2: Let $J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ a Jordan block.

$$\operatorname{Im}(J) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \quad \ker(J) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\operatorname{Im}(J^2) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}. \quad \text{and } J^3 = 0.$$

Idea:

- $\vec{z}_1 - \cancel{J\vec{z}_0}^{\,0} = \vec{w}_0$

  choose $\vec{z}_1 = \vec{w}_0$

- $2\vec{z}_2 - J\vec{z}_1 = \vec{w}_1 \rightsquigarrow$ let $\vec{z}_2 = \frac{1}{2}(J\vec{z}_1 + \vec{w}_1)$

- we have solved $\vec{z}_{k+1}$, then we want

  $(k+2)\vec{z}_{k+2} - J\vec{z}_{k+1} = 0. \Rightarrow \vec{z}_{k+2} \in \operatorname{Im}(J)$

- similarily, $\vec{z}_{k+3} \in \operatorname{Im}(J^2). \Rightarrow \boxed{J\vec{z}_{k+3} = 0}$

  i.e. it can be solved for $l = 3$.

Case 2: $\vec{r}(t) = e^{\alpha t} P_k(t)$, then we let $Y(t) = e^{\alpha t} Q_l(t)$.

$$\left(\frac{d}{dt} e^{\alpha t}\right)(Q_l(t)) = e^{\alpha t} A Q_l(t) + e^{\alpha t} P_k(t).$$

$$\rightsquigarrow \left(\frac{d}{dt} + \alpha I\right)(Q_l(t)) = A\, Q_l(t) + P_k(t).$$

$$\Longleftrightarrow \frac{d}{dt}(Q_l(t)) = (A - \alpha I) Q_l(t) + P_k(t).$$

§ Stability of solutions of 1^{st} order linear system:

- We consider $\vec{y}'(t) = A\vec{y}(t)$ $\quad \cdots\cdots \quad (*)$

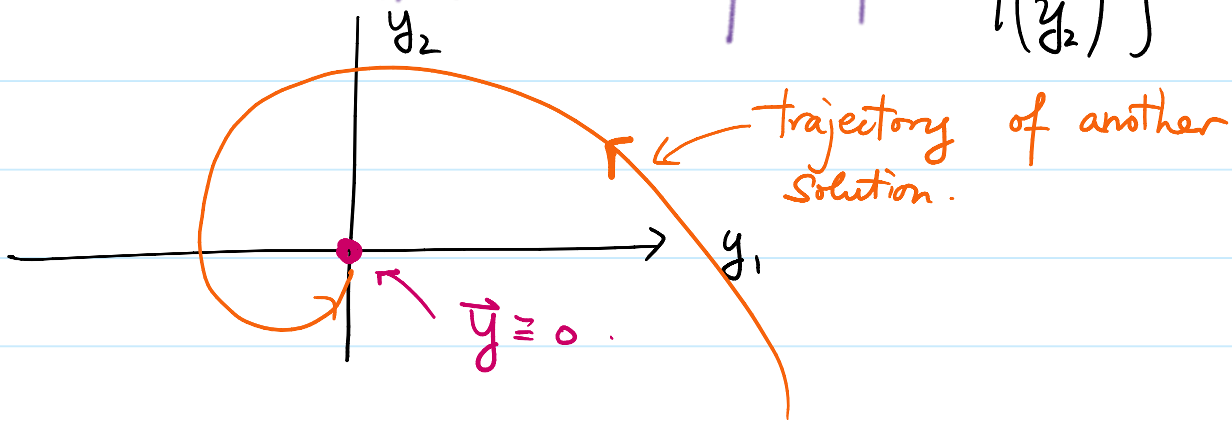
constant 2×2 matrix

Assume: $\det(A) \neq 0$, i.e. invertible

and we want to study the behaviour of solution for large t.

Observation: $\vec{y}(t) \equiv \vec{0}$ is the unique constant solution.

We will draw the solutions on the phase plane $= \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\}$



Def:
- We call the $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$-plane the phase plane,
- curve traced out by a solution $\vec{y}(t)$ a trajectory

- A representative set of trajectory a phase portrait.

meaning that it indicate all possible behaviour of different trajectory.

§ Linear system with two unknown: $A \in M_{2\times 2}(\mathbb{R})$.

There are three cases for eigenvalues:

1) $r_1 \neq r_2$ distinct real eigenvalues

2) $\lambda = \alpha + i\mu$, $\bar{\lambda} = \alpha - i\mu$ complex eigenvalues.

3) Repeated real eigenvalues r.

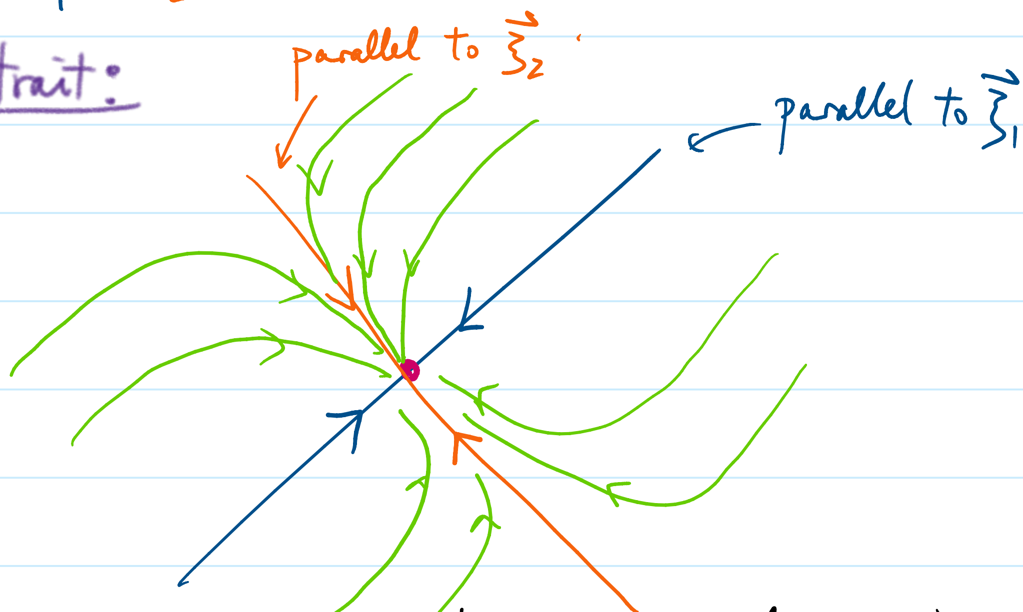
Case 1: $\vec{\xi}_i$ eigenvectors for r_i

General sol: $\vec{y}(t) = c_1 e^{r_1 t} \vec{\xi}_1 + c_2 e^{r_2 t} \vec{\xi}_2$

Three subcase:

Case 1 a) $r_1 < r_2 < 0$

phase portrait:



1. • if $c_2 = 0 \Rightarrow$ the solution lies on the line parallel to $\vec{\xi}_1$.

• $r_2 < 0 \Rightarrow$ the trajectory move towards $\vec{0}$ as $t \to \infty$

• the same for $c_1 = 0$.

2. if both $c_1, c_2 \neq 0$, we have to analyse their relation:

since $r_1 < r_2 < 0$, we may write

$$\vec{y}(t) = e^{r_2 t}\left(c_1 \vec{\xi}_1 e^{\overbrace{(r_1 - r_2)}^{-ve}t} + c_2 \vec{\xi}_2 \right)$$

this term is very small when t large.

- therefore the solution is "almost" tangent to the direction along $\vec{\xi}_2$.

3. for $t < 0$ small enough, we write

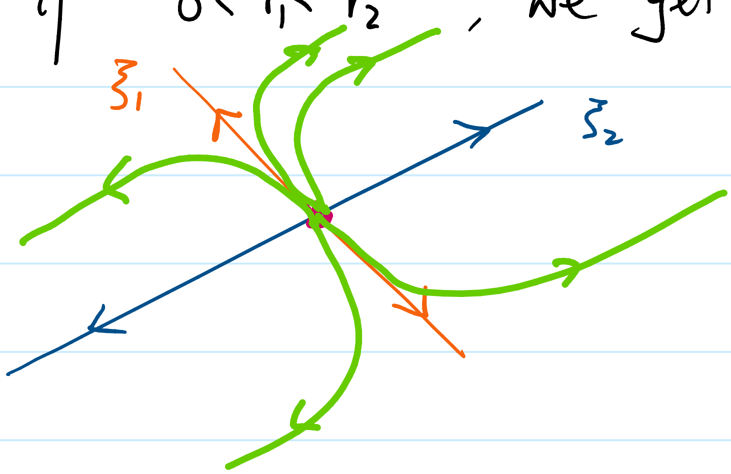
$$\vec{y}(t) = e^{r_1 t}\left(c_1 \vec{\xi}_1 + c_2 \vec{\xi}_2 e^{(r_2 - r_1)t} \right)$$

very small as $t \to -\infty$

$\therefore$ $\vec{y}(t)$ "almost" tangent to $c_1 \vec{\xi}_1$

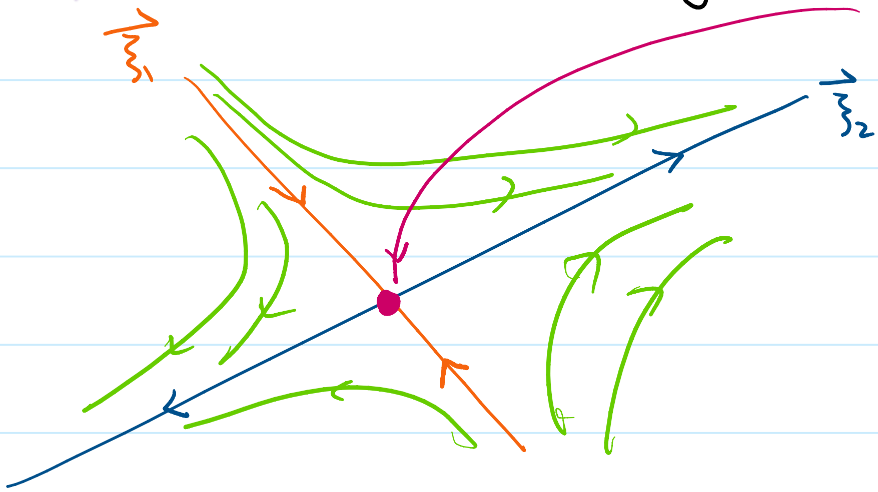
Conclusion:
- we have $\lim_{t \to \infty} \vec{y}(t) = 0$ for all the solution.
- $\vec{0}$ is called a nodal sink.

Case 1c: if $0 < r_1 < r_2$, we get a nodal source



Case 1b)

$r_1 < 0 < r_2$, we get a saddle



- by writing $\vec{y}(t) = e^{r_2 t}\left(c_1 \vec{\xi}_1 e^{(r_1 - r_2)t} + c_2 \vec{\xi}_2 \right)$

  Very small as $t \to \infty$.

  we have the curve tangent to $c_2 \vec{\xi}_2$ as $t \to \infty$.

- similarily, it tangent to $c_1 \vec{\xi}_1$ as $t \to -\infty$.

§ Case 2: Complex eigenvalue $\lambda = \alpha + i\mu$, $\bar{\lambda} = \alpha - i\mu$.

we have eigenvectors $\vec{\xi}_1 = \vec{u} + i\vec{v}$, $\vec{\xi}_2 = \vec{u} - i\vec{v}$.

General sol:

$$\vec{y}(t) = e^{\alpha t}\left(c_1 \vec{z}_1(t) + c_2 \vec{z}_2(t) \right)$$

where

$$\vec{z}_1(t) = \left(\vec{u} \cos\mu t - \vec{v} \sin\mu t \right)$$

$$\vec{z}_2(t) = \left(\vec{u} \sin\mu t + \vec{v} \cos\mu t \right).$$

Case 2a) For $\alpha = 0$:

We rewrite $c_1\vec{z}_1(t) + c_2\vec{z}_2(t)$

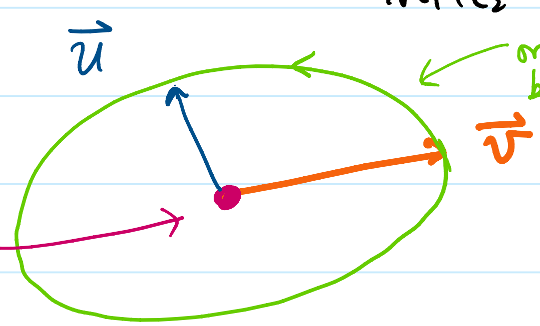
$$= \sqrt{c_1^2+c_2^2}\left(\vec{u} \cos\mu t \frac{c_1}{\sqrt{c_1^2+c_2^2}} + \vec{u} \sin\mu t \frac{c_2}{\sqrt{c_1^2+c_2^2}} \right)$$

$$+ \sqrt{c_1^2+c_2^2}\left(\vec{v} \cos\mu t \frac{c_2}{\sqrt{c_1^2+c_2^2}} - \vec{v} \sin\mu t \frac{c_1}{\sqrt{c_1^2+c_2^2}} \right)$$

$$= \sqrt{c_1^2+c_2^2}\left(\vec{u} \sin(\theta + \mu t) + \vec{v} \cos(\theta + \mu t) \right)$$

Where $\theta \in [0, 2\pi)$ s.t. $\sin(\theta) = \frac{c_1}{\sqrt{c_1^2+c_2^2}}$, $\cos(\theta) = \frac{c_2}{\sqrt{c_1^2+c_2^2}}$

phase portrait for $\mu > 0$:



center.

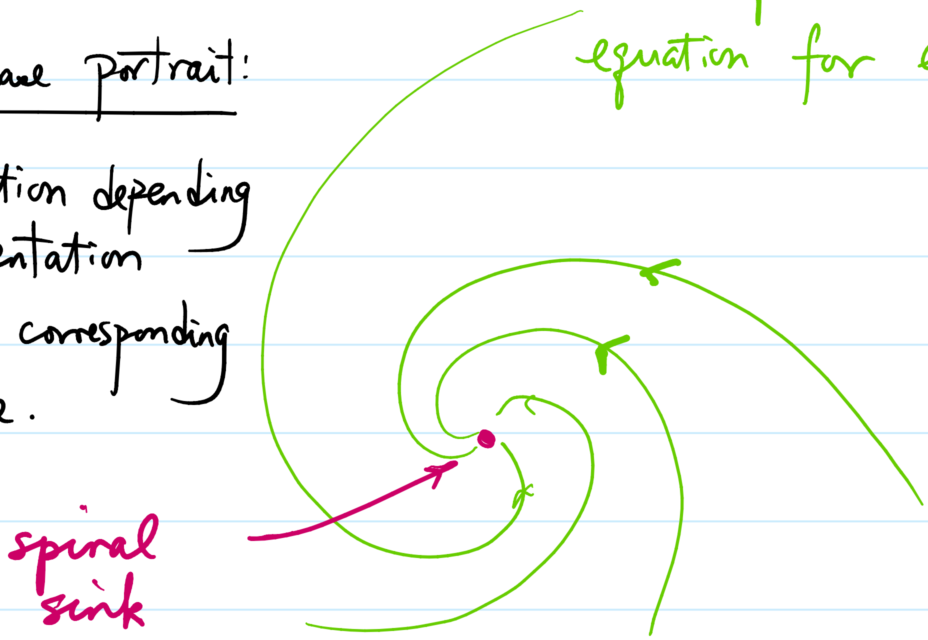
orientation determined by orientation of $\vec{u}, \vec{v}$ & sign of μ.

Case 2b) $\alpha < 0$, then we have

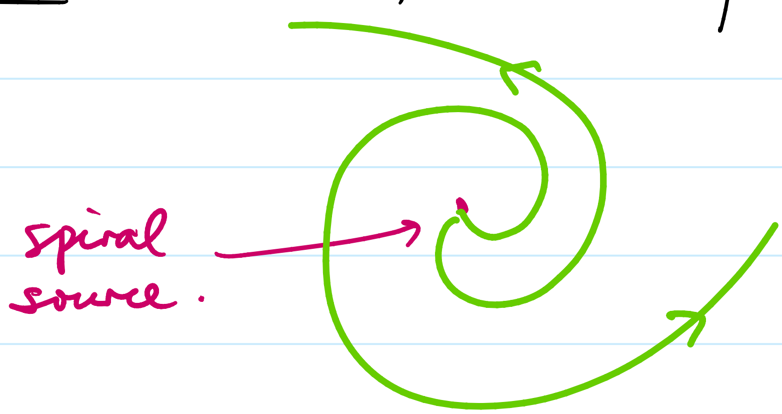
$$\vec{y}(t) = e^{\alpha t} \Big(\underbrace{c_1 \vec{\xi}_1(t) + c_2 \vec{\xi}_2(t)}_{\text{equation for ellipse.}} \Big)$$

phase portrait:

- orientation depending on orientation of the corresponding ellipse.



Case 2c) $\alpha > 0$, then it spiral outward.



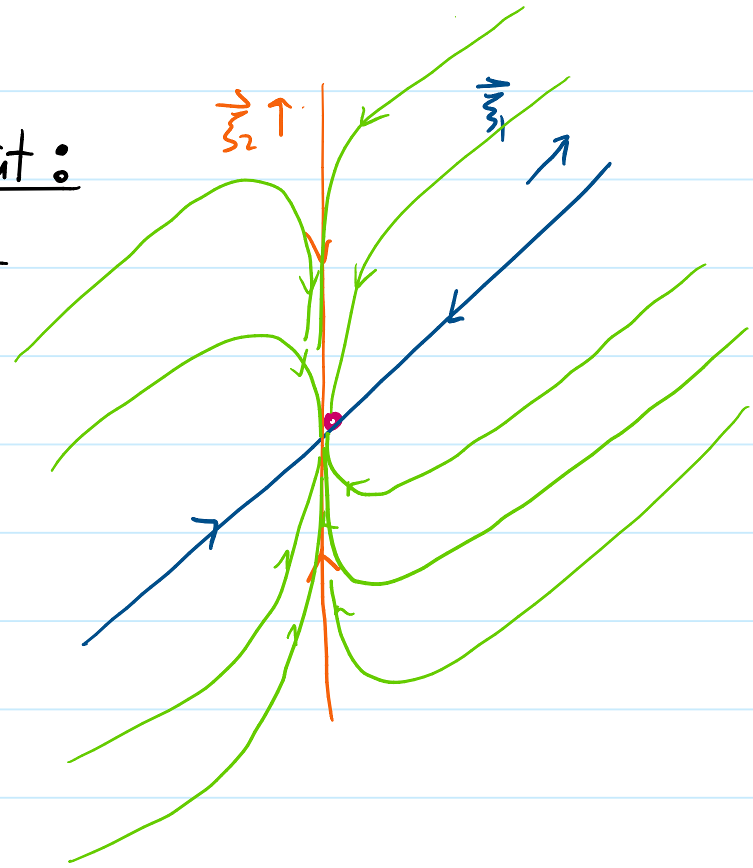
Eg. 1 $A = \begin{pmatrix} -2 & 0 \\ -2 & -\frac{1}{2} \end{pmatrix}$, $r_1 = -2$, $r_2 = -\frac{1}{2}$

with $\vec{\xi}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\vec{\xi}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

and we are in Case 1a)

phase portrait:

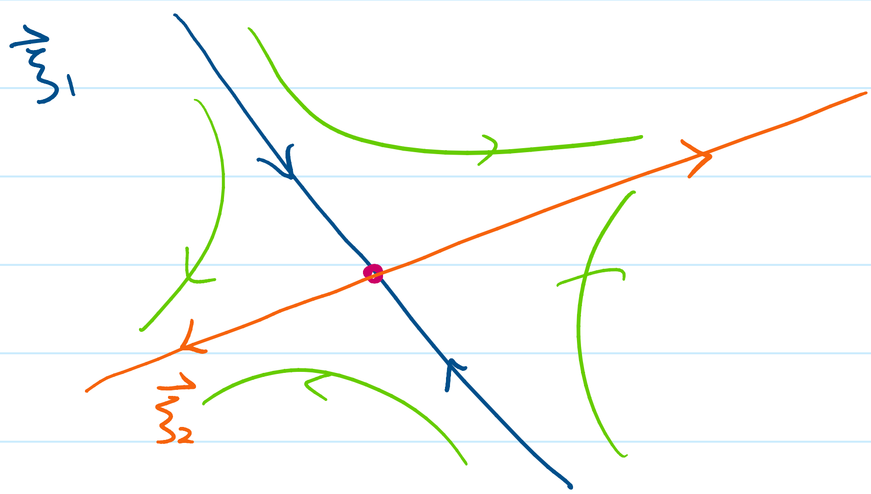
nodal sink:



Eg. 2 $\quad A = \begin{pmatrix} 3 & 2 \\ -2 & -2 \end{pmatrix} \sim r_1 = -1 < 0 < r_2 = 2$

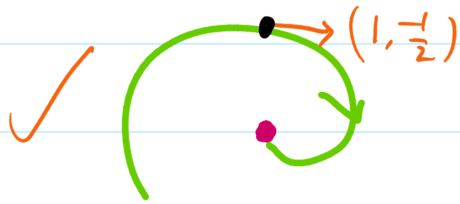
with $\vec{\xi}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\vec{\xi}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

and we are in case 1b ($\vec{0}$ is a saddle point)

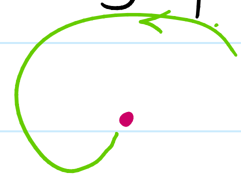


Eg. $A = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix}$, with complex eigenvector $\lambda = -1/2 \pm i$

We are in case 2b, with $\vec{0}$ being spiral source



or



In order to determine which case we are in:

Consider the solution $\vec{y}(t)$ passing through $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ at t_0

$$\rightsquigarrow \quad \vec{y}\,'(t_0) = A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}.$$

and hence we are in the first case.